THURSDAY, JULY 6, 1882

CLIFFORD'S MATHEMATICAL PAPERS

Mathematical Papers. By William Kingdon Clifford. Edited by Robert Tucker, with an Introduction by H. J. Stephen Smith. (London: Macmillan and Co., 1882.)

Mathematical Fragments; being Facsimiles of his Unfinished Papers Relating to the Theory of Graphs. By the late W. K. Clifford. (London: Macmillan and Co., 1881.)

ONLY those who wander much through the aridities of modern English mathematical text-books, whose duty compels them daily to read such literature, and who know

> "The mispent tyme, the service vaine,
> Whilk to considder is ane pane,"

can understand the pleasure of reviewing a book like Clifford's Papers. Here there is no occasion to yawn over page after page of commonplace, to mark with wonder the hundredth iteration of an ill-founded inference, to trace with languid amusement the method and arrangement of our ancestors, nay, the hereditary dots and dashes decrepid in the fourth generation. On the contrary, the novelty and variety alike of subject and of treatment is almost confusing, every page shadows forth some new idea, every line is informed with the personality and with the genius of its author.

Clifford was one of the bright spirits, all too few in number, who, in a generation, whose educational system is devoted to the encouragement of mediocrity and the cultivation of sciolism, saved the English school from the reproach of inability to follow their leaders. He was one of the select few who sat at the feet of Cayley and Sylvester, and shared their genius. When we compare him with the former of his great masters, he appears at first to want the steadfast purpose and rugged strength of our mathematical giant. The extreme, almost boyish vivacity of his style, and the refined elegance and studied variety of his methods give an impression of this kind which a nearer acquaintance with his work speedily dispels. Apart from his great originality, this elegance, popularity in the best sense, of style gave Clifford a specially important place among the leaders of the English School of Mathematicians, a place which there seems to be none left to fill. It was by his assistance that many were led to scale the almost inaccessible heights on which stand the structures of modern mathematics.

In some respects the exuberant philosophy of his popular works, especially his lectures, in which the more striking conclusions of modern mathematical science were presented to the uninitiated, must have harmed his reputation for solidity of thought. We are also inclined to doubt whether some of the enthusiastic non-mathematical souls that thought they had assimilated his teaching, really after all rose to the conception of Riemann's finite space of uniform positive curvature, in which the problem is solved of

> "Einer dem's zu Herzen ging
> Dass ihm sein Zopf so hinten hing,
> Der wollt' es anders haben."

Such a flight is given only to the sons of Genius, and to those who have in the first place painfully exercised their pinions in less ambitious journeys. Still these lectures of Clifford did good service in drawing the attention of the rising generation to the revolution that is taking place in the very elements of exact science. If every physical discovery of permanent or passing importance is to have its day in the drawing-room and the lecture-hall, why should the trumpet of mathematical progress not be blown occasionally in the streets of Gath and Ascalon? If too many be *called* in this way, some few may still be *chosen.* To these few the volume of Mathematical Papers will furnish the best help available in the English language to enable them to follow their calling. To our mind the popular lectures are cut too much after the passing fashion of the present day; and we should be surprised if the majority of those best qualified to judge of Clifford's work did not agree with us that it will be on the present volume that his future fame will rest. In our poor judgment there is ample foundation.

It would scarcely be proper here to criticise the papers in detail, with the view of pointing out the exact amount of originality in each. Besides, even if the reviewer felt more confident of his judgment in such a matter, the task were a needless one, for it has been done already, in the admirable introduction, by an authority to whom every English mathematician will at once bow.

The Papers have a somewhat fragmentary aspect. This is due in part to the immense range of Clifford's mathematical sympathy, which led him to write on a great variety of subjects; but mainly to the fact that many of the papers are actually unfinished, some of the most important being indeed mere sketches. Clifford seems to have cared, comparatively speaking, but little for the mere mathematical *Art;* his interest was reserved mainly for methods and principles. Accordingly we find him much occupied with new and far-reaching theories; and many of the memoirs in this volume are merely the outlines of vast schemes of work, which life and leisure were denied him to accomplish.

Besides advances in the Theory of Algebraic Integrals, the development of Projective Geometry, and the enormous extension of analysis that is included under the title of Higher Algebra, two great generalisations have marked the progress of modern mathematical science. The first of these is the extension of the axioms of geometry, which originated with Gauss, Bolyai, and Lobatschewsky, and was perfected by Riemann, and the theory of an n-fold manifoldness (Mannigfaltigkeitslehre) of which tridimensional geometry in this extended sense is only a particular case, Euclidian geometry a more particular case still. The second consists in a somewhat similar extension of the Axioms, or more strictly speaking, of the Laws of Operation, of Algebra, begun independently by Hamilton and Grassmann, and resulting in the first instance in the Quaternions of the one and the Ausdehnungslehre of the other. Both these generalisations have been progressive, and both appear to be pregnant with mighty results for the future. Clifford seized upon them with the instinct of genius. They pervade and colour the whole of his work, and the student who wishes clearly to understand the tendency of much that he has done must begin by attaining some mastery over these fundamental novelties. Great assistance will be obtained

from the elementary exposition of them given by Prof. Smith in his introduction to the Papers, pp. xl. *et seqq.* We should like, however, if we might venture to differ from so great an authority, to take exception to his definition of *flatness* by means of the notion of *planeness*, and to the introduction of the idea and the word *curvature* into an elementary exposition of the properties of space. This seems at best an explanation of the less by the more difficult; and, after all, the use in this case of the word *curvature* is of questionable propriety (although sanctioned by the highest authority), inasmuch as it suggests not only true but also false analogies. It is very well in the hands of a mathematician, to whom it suggests merely that a certain common apparatus of mathematical formulæ is applicable to a particular class of manifoldness and to a particular kind of surface; but to the mathematically untrained or half-trained reader the word suggests the paradox that portions of space on the two sides of a plane in elliptic space both are and are not congruent. Much harm has, we are persuaded, been done by this unfortunate usage of words. A similar piece of mystery making has been practised with n-dimensional space; the language of mathematicians concerning which has been retailed to ordinary simple-minded people as if it had the literal sense they naturally attach to it.

Clifford's papers on the geometry of hyper-space began with his translation of Riemann's famous Habilitationsschrift on the hypotheses which lie at the basis of geometry. He establishes a close connection between the generalised geometry and the generalised algebra in the Preliminary Sketch of Biquaternions, to our mind one of the ablest of his papers. He farther develops the subject in the memoirs "On the Motion of a Solid in Elliptic Space," "On the Theory of Screws in a Space of Constant Positive Curvature," "On the Free Motion under no forces of a Rigid System in an n-fold Homaloid." The kinematic of elliptic space as given by Clifford, and developed quite recently by Dr. Ball, forms one of the most elegant and attractive of modern geometrical theories. The starting point may be said to be the finding of the analogue in elliptic space to Euclid's parallel. In the modern geometrical sense a parallel (*i.e.* a line intersecting another at an infinite distance) cannot of course exist in elliptic space except as an imaginary line. If, however, we define a parallel as the straight equidistant from a given straight line, then through every point in space two parallels (a right and a left parallel as Clifford calls them) can be drawn to a given straight line. This appears at once by drawing at the given point a tangent plane to the equidistant surface of the given straight line, which it will be remembered is, in elliptic space, an anticlastic surface of revolution of the second degree, every zone of which is congruent with every other of the same breadth. This tangent plane meets the surface in two rectilinear generators, which intersect at the given point and have the property of equidistance from the given line. Parallels in this sense are of course imaginary in hyperbolic space, Euclid's parallel being the transition case for parallels in both senses. It seems a pity that a new word has not been used for this species of parallel.

It follows at once by synthetic reasoning of the simplest kind (in which we may in fact dispense with the aid of biquaternions or analytical aid of any kind) that almost all the properties of Euclidian parallels and parallelograms have their counterpart in the theory of Clifford's parallels, due attention being paid to the distinction of right and left. It is shown that a motion of a rigid body is possible in elliptic space such that every point moves in a right parallel, or every point in a left parallel, to a given straight line. A motion of the first kind is called a right vector, a motion of the second kind a left vector. The composition of two right vectors gives a right vector, and two left vectors a left vector; whereas the composition of a right vector with a left vector gives the most general motion of a rigid body, which Clifford calls a motor. It was to represent the ratio of two such displacements that Clifford invented his Biquaternion. Translation, strictly analogous to that in Euclidian space, *i.e.* rotation about the line at infinity does not exist in elliptic space. We may, of course, cause a body so to move that every point of it remains equidistant from a given line, and *in the same initial plane with that line.* Such a displacement is the same as a rotation about the polar of the given line, and is hence called by Clifford a Rotor. We have then the fundamental proposition, that every motor can be represented in an infinite number of ways as the sum of two rotors, but uniquely as the sum of two rotors whose axes are polar conjugates. It is the abolition of the line at infinity, whereby duality is made perfect, that gives the peculiar completeness and elegance to the properties of elliptic space, and fit it to be the paradise of geometers, where no proposition needs to wander disconsolate, bereft of its reciprocal.

To the second great branch of mathematical theory above alluded to, Clifford made exceedingly important contributions in his memoirs on the "Applications of Grassmann's Extensive Algebra," and "On the Classification of Geometric Algebras." Following, to some extent, in the footsteps of B. Peirce, whose epoch-making memoir has been given to the public at last in the *American Journal of Mathematics* for the current year, Clifford treats the subject with an incisive vigour all his own. The point of view (indicated by the word *geometric*) is no doubt limited, just as Peirce's is in another way; and there may be some doubt, as yet, as to the exact nature of the foundations upon which the reasoning rests. There is a lingering trace of the old sophistry in Peirce's work, here and there, at least so it appears to us; a reliance still upon ideas *a priori*, and a reluctance to abandon the restrictions imposed upon algebra by its arithmetical origin. Yet there can be no question as to the great value of the results already obtained and the immense extension of the mathematical horizon thereby effected. Already the attention of mathematical workers has been powerfully drawn to the matter, and there is hope that ere long another great theory equal in importance to the Mannigfaltigkeitslehre will drive its roots through the mathematical soil.

We have dwelt on two of the subjects touched upon in the "Papers," because they seem to us to be of the greatest immediate importance, and to show Clifford at his best as an original mathematician. But it must not be supposed that there is no other food for the mind mathematical in this volume. On the contrary, not one of these papers but is full of délight and edification, even

for the most highly educated reader. The charming simplicity of their style, the omission of everything like superfluous detail, and the great variety and importance of the subjects treated, will make the book an indispensable *vade mecum* for the tyro in pure mathematics. We think with regret of the infinite use it would have been to us in our learning years; from it we could have gathered, easily and pleasantly, in the pliant hours of youthful leisure, what we are now constrained to glean, in the intervals of ordinary drudgery, from partial treatises, and articles in foreign periodicals often the driest of the dry.

We must not conclude this notice without alluding to the appendix to the volume of papers, the most important parts of which are the fragment of a treatise called "The Algebraic Introduction to Elliptic Functions," the Notes of Clifford's Mathematical Lectures, and the problems and solutions contributed to the *Educational Times*. The fragment on elliptic functions, which deals with the Theta functions, has great value, as it gives a treatment of the subject not to be found in any English text-book. The lecture notes will be most useful to such teachers of mathematics as are sufficiently alive to the need of some modification of our traditional methods to take advantage of them. They remind us of the irreparable loss we have sustained by Clifford's early death of a doughty champion in the reformation of our degenerate system of mathematical education, which strangles the youthful mathematician ere he is born. It is, perhaps, too much to expect that the veteran chiefs of mathematical science should enter into the work of the drill-sergeant of mathematical recruits. They cannot be asked to devote their time to the petty work of reorganising the teaching of geometry and algebra in our schools and colleges. The more reason that we should mourn the departure of one who was able to take his place with the gods immortal, and yet disdained not to mingle with us mortals in the dusty fray of the Trojan Plain.

The handsome folio of lithographed manuscripts relating to the Theory of Graphs, forms one more monument of Clifford's genius, and affords us one more reason to lament our loss. Fully as we appreciate what he has actually done for us, and much as we are grateful for it, we cordially sympathise with the feeling that prompted the editor of the papers to put on the title-page the saying of Newton concerning Cotes: "If he had lived, we might have known something;" for, if we measure Clifford's promise by his actual performance, we see that he certainly died before his work was well begun.

G. CHRYSTAL

OUR BOOK SHELF

Winters Abroad. Some Information respecting Places visited by the Author on account of his own Health. Intended for the use of Invalids. By R. H. Otter, M.A. 8vo., pp. 236. (London: John Murray, 1882.)

THE places visited by the author are Australia, including Melbourne, Sydney, Queensland, and the Riverina, Tasmania, Algiers, Egypt, Cape of Good Hope, and Davos. He gives a short account of the places in the order in which he visited them, written in an easy readable style. The author's object in writing is to give invalids an idea of the easiest routes by which to reach health resorts, the kind of accommodation they may expect, the weather they must be prepared for, and the occupations and amusements which the several places afford. He has kept this object constantly before him, and has consequently produced a book which, notwithstanding its moderate size, clear, large type, and easy style, yet contains a great quantity of solid information which is quite trustworthy as far as it goes. From the nature of the work, embodying as it does the author's personal experience only, it is not complete, and might possibly mislead invalids who decided to follow the author without reference to the exact condition of their own lungs. For example, the author prefers the route to Egypt by the P. and O. steamers, and for many persons this may be excellent, but it involves a passage through the Bay of Biscay, with the possibility of rough weather, which to many invalids might be exceedingly dangerous. He has done good service to invalids by warning them of the necessity for warm clothing everywhere, but he speaks of throwing his coat on his bed as an additional covering, and so appears not to have had with him that greatest of all comforts to an invalid, an eider down quilt, which keeps him warm in bed, sofa, or chair, and when packed in a waterproof cover, is easily carried and serves at need for a pillow or footstool. In his remarks on Davos, the author observes, that through want of knowledge of the kind of cases for which the climate is suitable, many persons are sent there who would be much better elsewhere, and makes a most sensible suggestion that the authorities of Brompton Hospital should send thither a certain number of test cases. Proper accommodation and medical attendance would have to be provided for them, but the expense would not be very great, and might be met by special subscriptions for the purpose, while "the benefit to many of the sufferers and to the world at large might be incalculable." The origin of tubercle from germs, which has recently received such confirmation from Koch's experiments, as well as the increasing probability that under certain conditions these germs may be inoculated, afford a hope that consumption may ere long be brought, like typhoid fever, into the category of preventible diseases. But even after its causation is known as well as that of typhoid, cases will continue to occur from ignorance, stupidity, or negligence; but we may trust that it will no longer be the awful scourge which it is at present. To those who suffer from it, and who require to winter abroad, the present work will be a useful adviser and companion, and we would also strongly recommend its perusal to medical men who are personally unacquainted with the health-resorts to which they recommend their patients.

A Sequel to the First Six Books of the Elements of Euclid. By John Casey, LL.D., F.R.S. (Dublin: Hodges, 1882.)

THIS handy book has deservedly soon reached a second edition. In this way it will be seen that it has met a want. "The principles of modern geometry contained in the work are, in the present state of science, indispensable in Pure and Applied Mathematics and in Mathematical Physics; and it is important that the student should become early acquainted with them." The author appears to have thoroughly revised the text, and he has added many notes of interest, a few figures, we believe, and several problems: the enunciations occupy more space; that is, such terms as parallelogram are given in full, instead of being symbolically represented; but in the demonstrations the symbols are retained. An index has been added at the end.

We have noted a few errata: in the list of errata, for 4 read 74; p. 39, l. 15, "AB" should be "A C," as in first edition; on pp. 95, 157, the names of Sir W. Thomson and M. Mannheim are incorrectly printed; p. 110, the reference to *Educational Times* should be to the "reprint" from that journal; but these are very slight